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Calogero–Sutherland oscillator: classical behaviour and coherent states

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Abstract. Two kinds of coherent states are constructed in the context of the Calogero–Sutherland singular oscillator. The motion of the peaks of the wavefunctions of these coherent states are compared with the classical trajectory. It is found that while the wavefunction for one kind of coherent states is always singly peaked, that for the other acquires multiple peaks close to the classical turning point near the origin. The two coherent states are found to exhibit a kind of complementarity.

1. Introduction and summary

The Calogero–Sutherland model [1–2] describing a quantum system of N kinematically similar particles in one dimension, interacting pairwise via quadratic and centrifugal potentials, has attracted considerable interest in recent times. Not only does it provide an example of an exactly solvable many-body system, it has also been found relevant to the quantum Hall effect, fractional statistics and anyons [3–5]. For any quantum mechanical system, particularly an exactly solvable one, a natural question to ask is the one Schrödinger [6] raised and answered in the context of a harmonic oscillator: Which quantum mechanical states have the property that the peak of the modulus square of their wavefunctions follows the classical trajectory without changing form? As is well known, this search led to the notion of coherent states [7]. The purpose of this work is to examine this question in the context of the two-particle Calogero–Sutherland model, a problem, which, after removing the centre-of-mass motion, reduces to that of a singular oscillator described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 X^2 + \frac{g^2}{X^2}. \tag{1}$$

The eigenvalues and eigenfunctions for this Hamiltonian can be obtained using two methods. While the first [8] makes use of the su(1,1) algebra, the second [9–11] is based on factorization of a more general 'Hamiltonian' which is such that its eigenstates, suitably restricted, yield the eigenstates of the problem at hand. Both of these approaches have been used for an exact solution of the N-body Calogero–Sutherland model. We shall first briefly review the group theoretic approach and defer the discussion on the second method of solution to section 4.

The Hamiltonian operator

$$H = \frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}X^2} + \frac{1}{2}m\omega^2 X^2 + \frac{g^2}{X^2} \tag{2}$$

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on introducing the dimensionless variables

$$x = \left(\frac{m\omega}{\hbar}\right)^{1/2} X \qquad \mathcal{G}^2 = \frac{mg^2}{\hbar^2} \qquad \mathcal{H} = \frac{1}{\hbar\omega} H \tag{3}$$

can be written as

$$\mathcal{H} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 + \frac{\mathcal{G}^2}{x^2}. \tag{4}$$

Following Perelemov [8], we define

$$K_- = \frac{1}{2}\left[a_0^2 - \frac{\mathcal{G}^2}{x^2}\right] \qquad K_+ = \frac{1}{2}\left[a_0^{\dagger 2} - \frac{\mathcal{G}^2}{x^2}\right] \tag{5}$$

$$K_z = \frac{\mathcal{H}}{2} = \frac{1}{2}\left[a_0^\dagger a_0 + \frac{1}{2} + \frac{\mathcal{G}^2}{x^2}\right] \tag{6}$$

where

$$a_0 = \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right) \qquad a_0^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right) \qquad [a_0, a_0^\dagger] = 1. \tag{7}$$

It can easily be verified that these operators satisfy the su(1,1) algebra

$$[K_z, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_z. \tag{8}$$

From the su(1,1) structure, it follows that $\mathcal{H}$ has discrete eigenvalues E_n, $n = 0, 1, \ldots$ given by $E_n = 2n + E_0$ and the corresponding eigenstates ψ_n

$$\mathcal{H}\psi_n = E_n\psi_n \tag{9}$$

are given by

$$\psi_n = K_+^n \psi_0 \tag{10}$$

where ψ_0 satisfies

$$K_-\psi_0 = 0 \tag{11}$$

$$H\psi_0 = E_0\psi_0. \tag{12}$$

The explicit expressions for the normalized eigenfunctions of $\mathcal{H}$ are found to be

$$\psi_n(x) = (-1)^n \left[\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1/2)}\right]^{1/2} x^\alpha \mathrm{e}^{-x^2/2} L_n^{(\alpha-1/2)}(x^2) \tag{13}$$

where $\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{1+8\mathcal{G}^2}$ is the positive root of the equation $\alpha(\alpha-1) = 2\mathcal{G}^2$. These form an orthogonal set in the interval $(0, \infty)$. The action of $K_\pm$ on these states is given by

$$K_+\psi_n(x) = \sqrt{(n+1)(n+\alpha+1/2)}\,\psi_{n+1}(x) \tag{14}$$

$$K_-\psi_n(x) = \sqrt{n(n+\alpha-1/2)}\,\psi_{n-1}(x). \tag{15}$$

2. Coherent states

There are two kinds of coherent states which can be defined for the singular oscillator under consideration.

2.1. Eigenstates of K_-

For the ordinary harmonic oscillator, the ground state is uniquely given by the state which is annihilated by a_0. Coherent states, in this case, are taken to be the eigenstates of a_0. By analogy, for the singular oscillator one may define coherent states as the eigenstates of the operator K_-

$$K_-\Psi_1 = \lambda\Psi_1. \tag{16}$$

Putting

$$\Psi_1 = x^\alpha \exp(-x^2/2)\Phi \tag{17}$$

equation (16) may be written as

$$x^2\frac{\mathrm{d}^2}{\mathrm{d}^2x}\Phi + 2\alpha x\frac{\mathrm{d}}{\mathrm{d}x}\Phi - 4\lambda x^2\Phi = 0. \tag{18}$$

The solution of this equation is given by

$$\Phi(x,\lambda) = x^{(1/2-\alpha)}I_{(\alpha-1/2)}(2(\lambda x^2)^{1/2}). \tag{19}$$

The normalized eigenstates of K_- are thus given by

$$\Psi_1(x,\lambda) = \mathcal{N}[x^\alpha\exp(-x^2/2)][(\lambda x^2)^{-(\alpha-1/2)/2}I_{(\alpha-1/2)}(2(\lambda x^2)^{1/2})\mathrm{e}^{-\lambda}] \tag{20}$$

where

$$\mathcal{N} = \left[\frac{2}{I_{(\alpha-1/2)}(2|\lambda|)}\right]^{1/2}. \tag{21}$$

Rewriting (20) as

$$\Psi_1(x,\lambda) = \mathcal{N}[(\lambda)^{-(\alpha-1/2)/2}\mathrm{e}^{-\lambda}][x^{1/2}I_{(\alpha-1/2)}(2(\lambda x^2)^{1/2})\exp(-x^2/2)] \tag{22}$$

and using the fact that the expression in the second set of square brackets can be expanded in terms of Laguerre polynomials one obtains the following expansion for $\Psi_1(x,\lambda)$ in terms of the eigenstates of $\mathcal{H}$:

$$\Psi_1(x,\lambda) = \left[\frac{1}{I_{(\alpha-1/2)}(2|\lambda|)}\right]^{1/2}\sum_{n=0}^{\infty}\frac{(\lambda)^n}{\sqrt{\Gamma(n+1)\Gamma(n+\alpha+1/2)}}\psi_n(x). \tag{23}$$

The expectation value of $\mathcal{H}$ in the state $\Psi_1(x,\lambda)$ is found to be

$$\begin{aligned}\langle\mathcal{H}\rangle &= 2|\lambda|\left[\frac{I^{(2|\lambda|)}_{(\alpha-3/2)}}{I^{(2|\lambda|)}_{(\alpha-1/2)}}\right] - (\alpha-3/2)\\ &= |\lambda|\frac{\mathrm{d}}{\mathrm{d}}|\lambda|\log[I_{(\alpha-1/2)}(2|\lambda|)] + 1.\end{aligned} \tag{24}$$

For large z, using the asymptotic result

$$I_\nu(z) \cong \frac{e^z}{\sqrt{2\pi z}} \tag{25}$$

one finds that

$$\langle\mathcal{H}\rangle \cong 2|\lambda| + 1/2. \tag{26}$$

Further, from the commutation relations between K_- and $\mathcal{H}$, it follows that if the singular oscillator is initially prepared in the coherent state $\Psi_1(x, |\lambda|\mathrm{e}^{-\mathrm{i}\theta})$ then the state at a later time is $\Psi_1(x, |\lambda|\mathrm{e}^{-\mathrm{i}(\theta+2\omega t)})$.

The coherent states discussed above, being eigenstates of $K_- \equiv K_x - \mathrm{i}K_y$, constitute a class of minimum uncertainty states appropriate to the SU(1,1) algebra generated by K_x, K_y and K_z, i.e. for these states one has the relation $(\Delta K_x)(\Delta K_y) = |\langle K_z\rangle|/2$. Using (5)–(7) the expressions for K_x, K_y and K_z, in terms of x and $p \equiv -\mathrm{i}\partial/\partial x$, are found to be

$$K_x = (x^2 - p^2)/4 - \mathcal{G}^2/2x^2 \qquad K_y = -(xp + px)/2 \quad \text{and} \quad K_z = (x^2 + p^2)/4 + \mathcal{G}^2/2x^2.$$

2.2. *Eigenstate of the canonical conjugate of K_+*

In an earlier work [12], it was shown that, to every coherent state which is an eigenstate of an annihilation operator F, one can associate a coherent state which is dual to it. The construction of the dual coherent states involves the notion of an operator G which is the canonical conjugate of $F^\dagger$, i.e. an operator which satisfies $[G, F^\dagger] = 1$ on all states in an appropriate sector of the Fock space. In the present context, the canonical conjugate $\mathcal{K}_-$ of K_+ is easily found to be

$$\mathcal{K}_-[1/(K_z + k)]K_- \qquad \alpha + 1/2 = 2k \tag{27}$$

and one has

$$[\mathcal{K}, K_+] = 1. \tag{28}$$

The eigenstates $\Psi_2(x, \beta)$ of $\mathcal{K}_-$

$$\mathcal{K}_-\Psi_2(x, \beta) = \beta\Psi_2(x, \beta) \tag{29}$$

are easily found to be

$$\Psi_2(x, \beta) = (1 - |\beta|^2)^k \sum_{n=0}^{\infty} \frac{\sqrt{\Gamma(n + 2k)}}{\sqrt{\Gamma(n + 1)\Gamma(2k)}} \beta^n \psi_n(x) \qquad \alpha + 1/2 = 2k \tag{30}$$

$$= \left[\frac{(1 - |\beta|^2)}{(1 + \beta)^2}\right]^k \frac{\sqrt{2}}{\sqrt{\Gamma(2k)}} x^\alpha \exp\left[-\frac{1}{2}\frac{(1 - \beta)}{(1 + \beta)}x^2\right]. \tag{31}$$

In view of the fact that the canonical conjugate of $\mathcal{K}_-$ and K_+, these states are evidently the *states obtained by the application of the operator* $\exp(\beta K_+)$ *on the state* $\psi_0(x)$ *annihilated* by $\mathcal{K}_-$.

$$\Psi_2(x, \beta) = (1 - |\beta|^2)^k \exp(\beta K_+)\psi_0(x). \tag{32}$$

These states are easily seen to coincide with the coherent states constructed by Perelemov [8] for the present case though from a different point of view. They can also be regarded as the states annihilated by the operator $\tilde{K}_-$ where

$$\tilde{K}_- = \exp(\beta K_+)K_- \exp(-\beta K_+) = K_- - 2\beta K_z + \beta^2 K_+ \tag{33}$$

or, in view of (28), as the states annihilated by the operator $K_-\beta(K_z + k)$ (with k defined by (30)).

$$[K_- - \beta(K_z + k)]\Psi_2(x, \beta) = 0. \tag{34}$$

Further, from the commutation relations between $\mathcal{K}_-$ and $\mathcal{H}$, it follows that if the singular oscillator is initially prepared in the coherent state $\Psi_2(x, |\lambda|\mathrm{e}^{-\mathrm{i}\theta})$ then the state at a later time is $\Psi_2(x, |\lambda|\mathrm{e}^{-\mathrm{i}(\theta+2\omega t)})$.

3. Classical behaviour

If the Hamiltonian given by (1) is considered classically then one finds that the classical turning points are located at

$$X = \pm\left[\frac{E}{m\omega^2} \pm \left(\frac{E^2}{m^2\omega^4} - \frac{2g^2}{m\omega^2}\right)^{1/2}\right]^{1/2}. \tag{35}$$

Classical motion is thus possible only when

$$E \geqslant E_{\rm cl} = \sqrt{2m}g\omega. \tag{36}$$

The classical trajectory is given by

$$X^2(t) = \frac{1}{m\omega^2}\left[E + \sqrt{E^2 - E_0^{\rm cl2}}\cos\theta(t)\right] \tag{37}$$

where $\theta(t) = 2\omega t + \theta$.

4. Classical trajectories and coherent states

We first consider the coherent state $\Psi_2(x, \beta)$. The peak of

$$|\Psi_2(x,\beta)|^2 = \frac{2}{\Gamma(2k)}\left[\frac{(1-|\beta|^2)^2}{(1+\beta)^2(1+\beta^*)^2}\right]^k \exp\left[-\frac{(1-|\beta|^2)}{(1+\beta)(1+\beta^*)}x^2 + 2\alpha\log x\right] \tag{38}$$

is located at

$$x_{\rm p}^2 = \alpha\frac{(1+\beta)(1+\beta^*)}{(1-|\beta|^2)}. \tag{39}$$

Putting $\beta = |\beta|{\rm e}^{-i\theta}$

$$x_{\rm p}^2 = (E_0 - 1/2)\left[\frac{1+|\beta|^2}{1-|\beta|^2} + \frac{2|\beta|}{1-|\beta|^2}\cos\theta\right] \tag{40}$$

the expectation value of $\mathcal{H}$ in the coherent state $\Psi_2(x, \beta)$ is found to be

$$\frac{\langle\mathcal{H}\rangle}{E_0} = \frac{1+|\beta|^2}{1-|\beta|^2} \qquad E_0 \equiv 2k \equiv \alpha + 1/2. \tag{41}$$

Using this relation in the expression for $x_{\rm p}^2$ to eliminate $|\beta|$ in favour of $\langle\mathcal{H}\rangle$ one obtains

$$x_{\rm p}^2 = \left(\frac{E_0 - 1/2}{E_0}\right)\left[\langle\mathcal{H}\rangle + \sqrt{\langle\mathcal{H}\rangle^2 - E_0^2}\cos\theta\right]. \tag{42}$$

The position of the peak changes according to the following equation

$$x_{\rm p}^2 = \left(\frac{E_0 - 1/2}{E_0}\right)\left[\langle\mathcal{H}\rangle + \sqrt{\langle\mathcal{H}\rangle^2 - E_0^2}\cos(2\omega t + \theta)\right]. \tag{43}$$

For large g, $E_0 \cong \sqrt{2\mathcal{G}} \geqslant 1/2$ yields the classical trajectory.

For the coherent $\Psi_1(x, \lambda)$, it is not possible to give an analytical expression for the position of the peak. Figures 1 and 2 show $|\Psi_1(x,\lambda)|^2$ and $|\Psi_2(x,\beta)|^2$ for various values of $|\lambda|$, θ, and α. For comparison with the coherent states $\Psi_2(x, \beta)$ we have chosen $|\lambda|$ so that the expectation value of $\mathcal{H}$ in the two cases are the same. The arrow on these figures indicates the classical position of the oscillator. Some salient features of these plots are as follows.

(i) At $\theta(0) = 0$ the function $|\Psi_1(x,\lambda)|^2$ is sharply peaked. The function $|\Psi_2(x,\beta)|^2$, on the other hand, is quite broad. As $\theta(t)$ increases, the width of $|\Psi_1(x,\lambda)|^2$ increases whereas that of $|\Psi_2(x,\beta)|^2$ decreases. This continues until $\theta(t)$ becomes close to π. The function $|\Psi_1(x,\lambda)|^2$ no longer remains single peaked (figure 1(*c*)). In contrast to this, the function $|\Psi_2(x,\beta)|^2$ continues to have a single peak and the peak is at its sharpest for $\theta(t) = \pi$. The two coherent states thus exhibit a kind of complementarity.

(ii) For large $\mathcal{G}$, the peak of $|\Psi_2(x,\beta)|^2$ follows the classical trajectory as is also evident from the analytical expression (43) for the position of the peak. The peak of the function $|\Psi_1(x,\lambda)|^2$, on the other hand, follows the classical trajectory for large $\langle\mathcal{H}\rangle$. (For values of θ close to π where the function $|\Psi_1(x,\lambda)|^2$ has many peaks, we consider the highest peak for the purpose of comparison with the classical trajectory.)

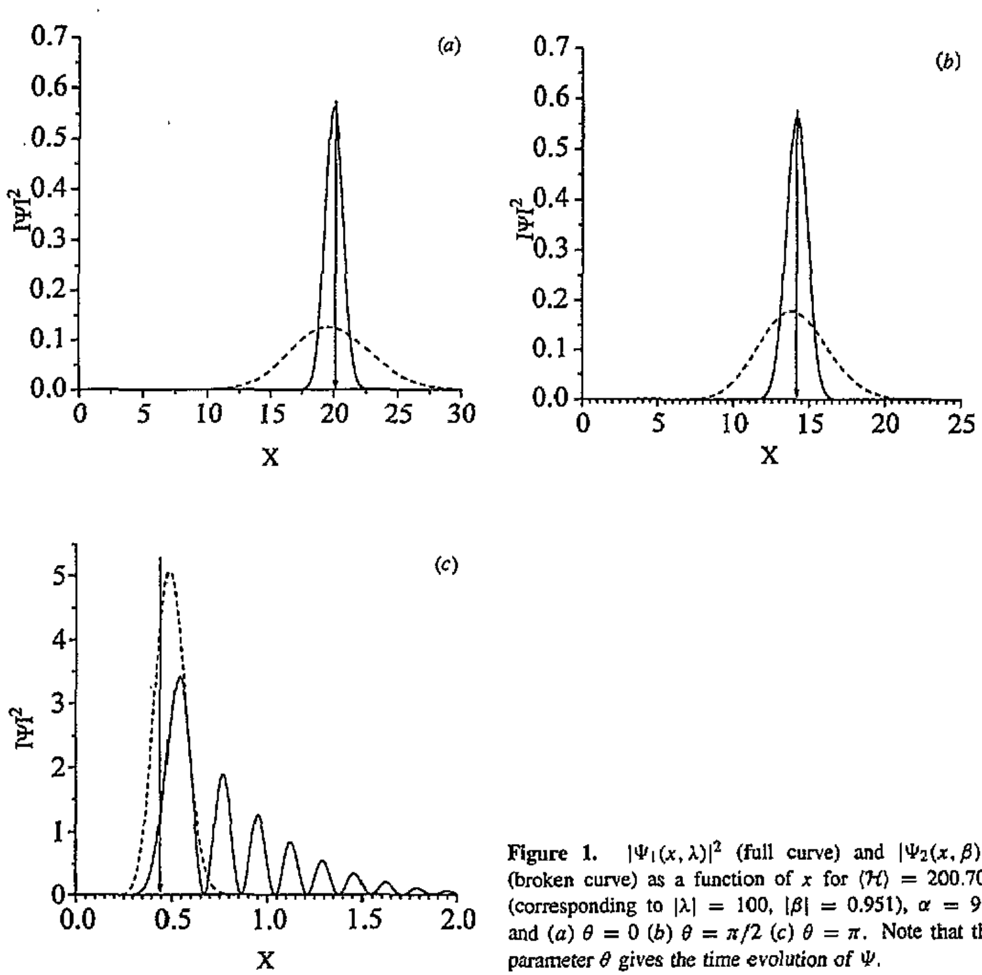

Figure 1. $|\Psi_1(x,\lambda)|^2$ (full curve) and $|\Psi_2(x,\beta)|^2$ (broken curve) as a function of x for $\langle\mathcal{H}\rangle = 200.702$ (corresponding to $|\lambda| = 100$, $|\beta| = 0.951$), $\alpha = 9.5$ and (*a*) $\theta = 0$ (*b*) $\theta = \pi/2$ (*c*) $\theta = \pi$. Note that the parameter θ gives the time evolution of Ψ.

5. Algebraic construction of the coherent states of the Calogero–Sutherland oscillator

In this section we briefly outline the connection between the coherent states studied above and the coherent states that may be defined in the context of the factorization method [9–11]

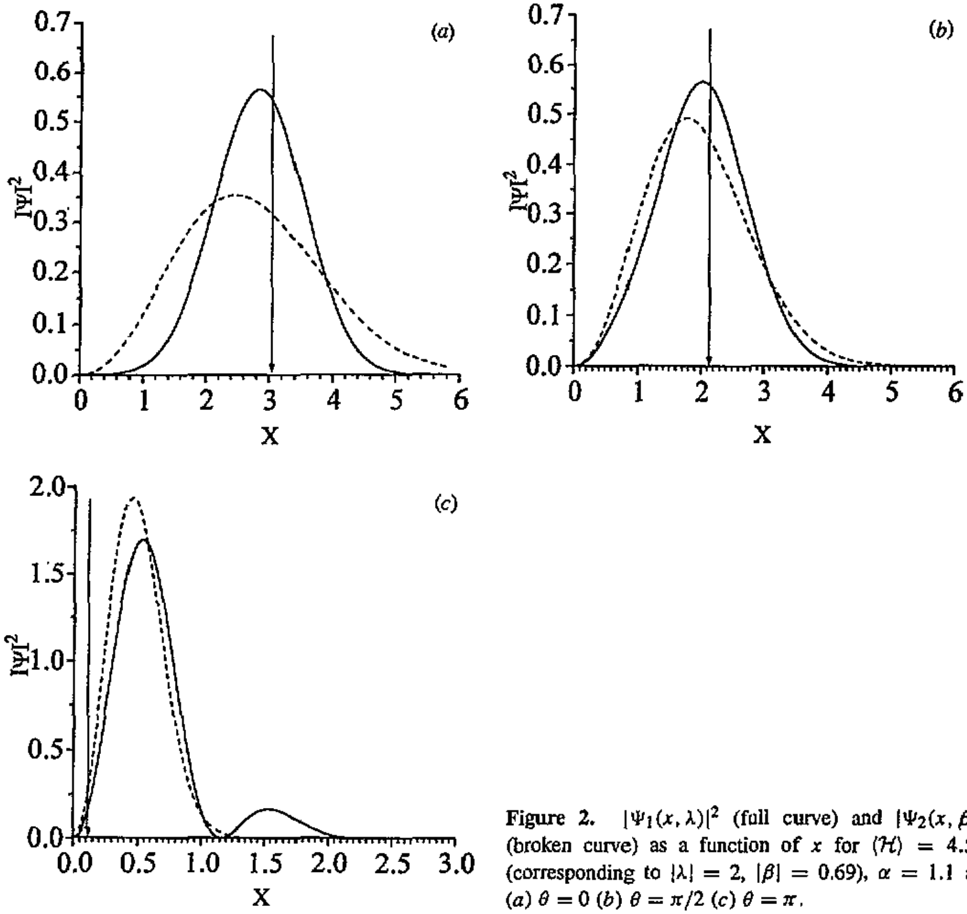

Figure 2. $|\Psi_1(x,\lambda)|^2$ (full curve) and $|\Psi_2(x,\beta)|^2$ (broken curve) as a function of x for $\langle\mathcal{H}\rangle = 4.522$ (corresponding to $|\lambda| = 2$, $|\beta| = 0.69$), $\alpha = 1.1$ and (*a*) $\theta = 0$ (*b*) $\theta = \pi/2$ (*c*) $\theta = \pi$.

for solving the Calogero–Sutherland model. In this method, instead of $\mathcal{H}$ given by (4), it proves convenient to work with the operator $\mathcal{H}'$ defined as follows

$$\mathcal{H}' = x^{-\alpha}\mathcal{H}x^{\alpha} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\alpha}{x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}x^2. \tag{44}$$

One then considers a more general operator

$$H' = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\alpha}{x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}x^2 + \frac{\alpha}{x^2}(1-M) \tag{45}$$

which differs from $\mathcal{H}'$ by an extra term involving the 'parity operator' M satisfying

$$M^2 = 1 \qquad \{M, x\} = \left[M, \frac{\mathrm{d}}{\mathrm{d}x}\right] = 0. \tag{46}$$

The added term has the advantage that the operator H' can now be written as

$$H' = \tfrac{1}{2}\{a_-, a_+\} \tag{47}$$

where

$$a_- = a_0 + \frac{\alpha}{\sqrt{2}_x}(1-M) \tag{48}$$

$$a_+ = a_0^\dagger - \frac{\alpha}{\sqrt{2}_x}(1-M). \tag{49}$$

The operators a_- and a_+ obey the following commutation relations:

$$[a_-, a_+] = 1 + 2\alpha M \qquad [a, h'] = a \qquad [a_+, h'] = -a_+. \tag{50}$$

Further, it is evident from (45) that the eigenstates of H' satisfying $M\phi_n(x) = \phi_n(x)$ are eigenstates $\phi_n(x)$ of $\mathcal{H}'$. Such states will be hereafter referred to as the states in the even sector. Now, by virtue of the commutation relations (50), the eigenstates of H' are given by

$$\phi_n(x) = a_+^n \phi_0 \tag{51}$$

where ϕ_0 satisfies

$$a_-\phi_0 = 0 \qquad \text{i.e.} \left[a_0 + \frac{\alpha}{\sqrt{2}x}(1 - M)\right]\phi_0 = 0. \tag{52}$$

Imposing $M\phi_n(x) = \phi_n(x)$, one finds that

$$\phi_0(x) = \exp(-x^2/2)$$
$$\phi_n(x) = a_+^{2n}\phi_0 = \left[\left[a_0^\dagger - \frac{2\alpha}{\sqrt{2}x}\right]a_0^\dagger\right]^n \exp(-x^2/2) \tag{53}$$

which, after multiplication by x^α, are easily found to be identical to the eigenfunctions (13) obtained using the group theoretic approach.

It is interesting to note that, in this method of solution, the coherent states $\Psi_1(x, \lambda)$ discussed earlier corresponds to the eigenstates of a_-^2 in the even sector. Likewise the coherent states $\Psi_2(x, \beta)$ correspond to the states annihilated by $a_- + \beta a_+$ in the even sector. The two coherent states thus bear the same relationship to each other as do the cat and the squeezed states in the context of the ordinary harmonic oscillator.

6. Conclusions

In this work we have considered two kinds of coherent state for the Calogero–Sutherland singular oscillator with a view to examining the question as to what extent the peaks of their wavefunctions follow the classical trajectory. It is found that, while one reproduces the classical motion for large values of the parameter g, the other gives a fairly good agreement with the classical behaviour for large values of the expectation value $\langle\mathcal{H}\rangle$ in that state. The two coherent states are not only duals of each other in the sense defined in [12] but also exhibit a kind of complementarity—in the regions where one coherent state is sharply peaked the other exhibits a broad peak and *vice versa*. Of the two coherent states considered one finds that while one is always singly peaked, the other breaks into several peaks close to the barrier at the origin. The relationship of these coherent states to those that can be constructed in the context of the algebraic method of solution of the Calogero–Sutherland oscillator is also brought out. Once the algebraic structure is clear one can construct other types of state as well, for example, the ones associated with linear combinations [13] of a_-^2 and a_+^2 which would belong to the general class of minimum uncertainty states associated with the singular oscillator.

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